Exercise Sheet I

Random Fields and Covariance

1. a. Let $\omega > 0$ be fixed and let $A, B \sim F$ independently with mean zero, finite variance and nonzero but finite third moment. Define $X(s) := A\cos(\omega s) + B\sin(\omega s)$ on $\mathbb{R}^1$.

i. Compute $\mathbb{E}(X(s))$ and $\mathbb{C}(X(s), X(s'))$, and show that X is weakly stationary.

ii. Show that X is not strictly stationary.

iii. What is a necessary and sufficient condition on F for X to exhibit strict stationarity?

b. Let $B : \mathbb{R}^+ \to \mathbb{R}$ be standard Brownian motion, ie $B(0) = 0$ almost surely, $B(s) - B(s') \sim \text{Gaussian}(0, s - s')$ for $s > s' \geq 0$ and increments over disjoint intervals are independent.

i. Show that B is intrinsically stationary. What is its variogram?

ii. Show that B is not weakly stationary. What is its covariance function?

iii. Define $X(s) := B(s) - B(s+1)$. Show that this process is weakly stationary. What is its covariogram?

2. Compute the spectral densities of the following proposed covariance functions on $\mathbb{R}^1$, and deduce that they are valid covariograms.

a. The generalised exponential covariogram: $\sigma_1(h) := \sigma_0 \exp(-|h|/\phi)$.

b. The Gaussian covariogram: $\sigma_2(h) := \exp(-h^2)$.

c. The Matérn covariogram with $\nu = 3/2$: $\sigma_3(h) := (1 + |h|)\exp(-|h|)$.

3. a. Prove that, for σ a valid covariogram with spectral density f in $\mathbb{R}^d$, $\sigma_\phi(h) := \sigma(h/\phi)$ has spectral density $f_\phi(\omega) = \phi^d f(\phi\omega)$, and deduce that σ_ϕ is a valid covariogram for all $\phi > 0$.

b. i. Let σ be a valid covariogram in $\mathbb{R}^{d_1}$. Show that $\sigma^*(h_1, h_2) := \sigma(h_1)$ for $h_1 \in \mathbb{R}^{d_1}$ and $h_2 \in \mathbb{R}^{d_2}$ is a valid covariogram in $\mathbb{R}^{d_1+d_2}$.

ii. Suppose that σ_1 and σ_2 are valid covariograms in $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$, respectively. Conclude that $\sigma(h_1, h_2) := \sigma_1(h_1)\sigma_2(h_2)$ for $h_1 \in \mathbb{R}^{d_1}$ and $h_2 \in \mathbb{R}^{d_2}$ is a valid covariogram in $\mathbb{R}^{d_1+d_2}$.

4. A random field has covariogram $\sigma(h) := \tau\mathbb{I}(h = 0) + \sigma_0 \exp(-\|h\|/\phi)$, where τ is the nugget, σ_0 is the semisill, and ϕ is the range parameter.

a. Compute the variogram $\gamma(h)$ for $\|h\| > 0$.

b. What is the sill? Compute the practical range, ie the lag $\|h^*\|$ where $\gamma(h^*) = 0.95\gamma(\infty)$.

c. Sketch $\gamma(h)$ for $\|h\| \geq 0$, labelling the nugget, semisill, sill and practical range.

5. Consider the proposed tent covariogram, $\sigma(h) = \max\{0, 1 - \|h\|\}$.

a. Demonstrate that this is a valid covariogram in $\mathbb{R}^1$, by computing its spectral density.

b. Show that its spectral density in $\mathbb{R}^2$ is given by: $f(\omega) = \frac{1}{2\pi}\int_0^1 (1-h)J_0(\omega h)h\mathrm{d}h$.

c. Using software, plot $f(\omega)$ and identify the first region in which it takes negative values. Deduce that the tent covariogram is not valid in any dimensionality greater than one.